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Markovian Solutions of Inviscid Burgers Equation

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For solutions of (inviscid, forceless, one dimensional) Burgers equation with random initial condition, it is heuristically shown that a stationary Feller–Markov property (with respect to the space variable) at some time is conserved at later times, and an evolution equation is derived for the infinitesimal generator. Previously known explicit solutions such as Frachebourg–Martin's (white noise initial velocity) and Carraro–Duchon's Lévy process intrinsic-statistical solutions (including Brownian initial velocity) are recovered as special cases.

KEY WORDS: Burgers; inviscid; turbulence; Markov.

1. INTRODUCTION

We consider the inviscid Burgers equation $\partial_t u + \partial_x(\frac{1}{2}u^2) = 0$ with random initial data u_0. Burgers equation has originally been introduced as a 1D model of turbulence. Although it is now clear that it does not exhibit lots of features of "true" turbulence, we nevertheless still think it is a good equation on which one can try and find new methods to apply on Euler equation. Having this in mind, taking random initial data seems quite a natural problem. It is also physically relevant in the contexts of interface dynamics, of aggregation of particles,[(7)] and some others. Burgers equation with a random force on the r.h.s. has also been studied, mainly as a "benchmark" to test methods designed for (Navier–Stokes) forced turbulence, many of which turn out to produce spurious predictions when applied to the simpler Burgers case. See ref. 6 for instance (where the hierarchy of evolution equations for the n-point densities is derived, including an anomaly term due to the presence of shocks).

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In the forceless case, usually (i.e., with u_0 rough enough, such as Brownian or white noise, or fractionary Brownian or its derivative, etc.) the solution at time t has a typical "random sawtooth" profile, with slope $1/t$ everywhere except on a denumerable set of points where negative jumps (shocks) occur. These jumps move and decrease and eventually coalesce to form bigger jumps, and the issue here addressed is to describe the statistics of $u(t, \cdot)$ for fixed time t, and this is essentially the probability distribution of the shocks' locations and sizes.

The case of a Brownian initial data has already been investigated by Sinai.[12] Carraro and Duchon[4,5] showed that Lévy processes are conserved by Burgers equation. They also obtained the explicit evolution equation for the characteristic function of the Lévy process solutions of Burgers. A noticeable point is that they made no use of the Hopf–Cole construction of the solution (Bertoin[2] recovered essentially the same result with Hopf–Cole).

The case of a white noise initial velocity, first considered by Jan Burgers himself,[3] was further analyzed by Avellaneda and E[1] who first observed the Markov property of u at later times (a consequence of Millar's theorem on Brownian process with a parabolic drift). This made it possible for Frachebourg and Martin to do the final step in making the (single time) statistics of the solution fully explicit.

Since Lévy processes are Markov, the question arises whether the property of being conserved by Burgers equation is true for more general Markov processes (for example, an Orstein–Uhlenbeck initial velocity).

Our investigation goes as follows: we consider statistical solutions of Burgers equation (otherwise called solutions of Hopf equation), and write an infinite set of equations for the n-point functions of such solutions, which parallels that established in ref. 6 (but using smooth test functions and moments instead of densities, we avoid the necessity of a separate anomaly term due to the presence of jumps). We show that the assumption that the process is Feller (in space) for all time yields an evolution equation for the infinitesimal generator of this process. Conversely, a Feller process whose generator satisfies this equation is a statistical solution of Burgers equation. This will allow us to recover Carraro and Duchon's result on Lévy processes, as a special case. Frachebourg and Martin's explicit solution[8] in the case of an initial white noise velocity is also a particular solution to our equation.

2. NOTATIONS AND DEFINITIONS

A Markov process $u(x)_{x \in \mathbf{R}}$ can be characterized by its one point and its transition probabilities $p_x(du)$ and $q_{x,y}(u, dv)$, $x < y$, that satisfy, $\forall x_0 < \cdots < x_k$ and f_i borelian positive,

$$\mathbf{E}\left[\prod_{i=0}^{k} f_i(u(x_i))\right]$$

$$= \int p_{x_0}(du_0)\, f_0(u_0) \int q_{x_0, x_1}(u_0, du_1)\, f_1(u_1) \cdots \int q_{x_{k-1}, x_k}(u_{k-1}, du_k)\, f_k(u_k).$$

A Markov process is *homogeneous* if its transition probabilities $q_{x,y}$ depend on x and y only through $y-x$. In this case, we write q_h instead of $q_{x,x+h}$.

A process $u(x)_{x \in \mathbf{R}}$ is *stationary* if and only if it is translation invariant: the law of $(u_{x+x_1}, ..., u_{x+x_n})$ does not depend on x. Hence a Markov process is stationary if and only if it is homogeneous and its one point probability $p_x(du)$ does not depend on x.

If u is a homogeneous Markov process, $h > 0$, and f is a continuous function vanishing at infinity, we put $Q_h f(u) = \int f(v)\, q_h(u, dv)$.

A *Feller* process is a homogeneous Markov process such that for each f, for each $h > 0$, $Q_h f$ is also continuous and vanishes at infinity, and $\lim_{h \to 0} Q_h f = f$ pointwise.

A Feller process always has a càdlàg version.[10]

One can define the *infinitesimal generator* of a Feller process: it is the operator A, defined for all the functions f such that the limit below exists, by

$$\forall x \in \mathbf{R}, \qquad Af(x) = \lim_{h \to 0^+} \frac{Q_h f(x) - f(x)}{h}.$$

Formally, $Q_h = \exp(hA)$, $Q'_h := dQ/dh = AQ_h$, and an invariant measure p_0 satisfies ${}^t A p_0 = 0$.

3. STATISTICAL SOLUTIONS OF BURGERS EQUATION

We will closely follow ref. 5 (see also ref. 11). Let E be the space of càdlàg real functions equipped with the smallest σ-algebra $\mathscr{C}(E)$ such that for each $x \in \mathbf{R}$, $u \mapsto u(x)$ is measurable. Let $\mathscr{D}$ be the set of real C^∞ functions with compact support. A probability μ on E is then characterized by its characteristic function

$$v \in \mathscr{D} \mapsto \int_E \exp\left[i \int_{\mathbf{R}} u(x)\, v(x)\, dx\right] d\mu(u) = \hat{\mu}(v).$$

Let u_0: $(\Omega, \mathscr{A}, P) \to E$ be a random process, defined on some probability space, and let μ_0: $\mathscr{C}(E) \to [0, 1]$ denote its probability law. Assume $u(x, t)$

is a (weak) solution of Burgers equation with $u(.,0)=u_0$, $u(.,t)\in E$ for $t>0$, and everything makes sense in the following calculation: integrability, and differentiability with respect to t. Let μ_t denote the law of $u(.,t)$. Formally, one then gets for each $v\in\mathscr{D}$:

$$\begin{aligned}
\partial_t\hat{\mu}_t(v) &= \int_E \partial_t\left\{\exp\left[i\int_{\mathbf{R}} u(x)\,v(x)\,dx\right]\right\}d\mu_t(u)\\
&= \int_E \partial_t\left\{\exp\left[i\int_{\mathbf{R}} u(x,t)\,v(x)\,dx\right]\right\}d\mu_0(u_0)\\
&= \int_E \exp\left[i\int_{\mathbf{R}} u(x,t)\,v(x)\,dx)\right]\partial_t\left[i\int_{\mathbf{R}} u(x,t)\,v(x)\,dx\right]d\mu_0(u_0)\\
&= \int_E \exp\left[i\int_{\mathbf{R}} u(x,t)\,v(x)\,dx\right] i\int_{\mathbf{R}} \tfrac{1}{2}u(x,t)^2\,v'(x)\,dx\,d\mu_0(u_0)\\
&= i\int_E\int_{\mathbf{R}} \tfrac{1}{2}u(x)^2\,v'(x)\,dx\exp\left[i\int_{\mathbf{R}} uv\right]d\mu_t(u).
\end{aligned}$$

Hence our definition of a *statistical solution* of Burgers equation:

Definition 1. A statistical solution of Burgers equation is a set $(\mu_t)_{t\geqslant 0}$ of probabilities on $(E,\mathscr{C}(E))$ such that for any $v\in\mathscr{D}$,

$$\partial_t\hat{\mu}_t(v) = i\int_E\int_{\mathbf{R}} \tfrac{1}{2}u(x)^2\,v'(x)\,dx\exp\left[i\int_{\mathbf{R}} uv\right]d\mu_t(u). \tag{1}$$

Let us assume now that we have a statistical solution of Burgers equation, $(\mu_t)_{t\geqslant 0}$, and that for all t, all the moments of μ_t are well defined. Then one can write

$$\exp\left[i\int_{\mathbf{R}} u(x)\,v(x)\,dx\right] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbf{R}^n}\prod_{j=1}^{n} u(x_j)\,v(x_j)\,dx_j.$$

Equation (1) thus becomes $\forall v\in\mathscr{D}$,

$$\begin{aligned}
&2\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbf{R}^n}\partial_t\mathbf{E}\left[\prod_{j=1}^{n} u(x_j)\,v(x_j)\right]\prod dx_j\\
&\quad= i\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{\mathbf{R}^{n+1}}\mathbf{E}\left[\prod_{j=1}^{n} u(x_j)\,v(x_j)\,u(x)^2\,v'(x)\right]dx\prod dx_j\\
&\quad= i\sum_{n=0}^{\infty} i^n\int_{x_0<x_1<\cdots<x_n}\sum_{j=0}^{n}\mathbf{E}\left[u(x_j)\frac{v'(x_j)}{v(x_j)}\prod_{k=0}^{n} u(x_k)\,v(x_k)\right]\prod dx_k.
\end{aligned} \tag{2}$$

4. EVOLUTION EQUATION FOR MARKOV SOLUTIONS

We are now looking for solutions such that at each time t, $x \mapsto u(x, t)$ is a stationary Feller process (with respect to space x). We are going to show that for such processes, the infinite set of Eqs. (2) is equivalent to an evolution equation for the infinitesimal generator of u.

We thus assume now that the solution $x \mapsto u(x, t)$ is a stationary Feller process, with one point probability $p(du, t)$ and transition probability $q_h(u_1, du_2, t)$, Eq. (2) becomes $\forall v \in \mathscr{D}$:

$$
\begin{aligned}
&2 \sum_{n=1}^{\infty} \int_{x_1 < \cdots < x_n} dx_1 \cdots dx_n \, u_1 v(x_1) \cdots u_n v(x_n) \\
&\quad \times \partial_t [\, p(du_1)\, q_{h_2}(u_1, du_2) \cdots q_{h_n}(u_{n-1}, du_n)] \\
&\quad = i \sum_{n=1}^{\infty} i^n \int_{x_0 < \cdots < x_n} dx_0 \cdots dx_n \, p(du_0)\, q_{h_1}(u_0, du_1) \cdots q_{h_n}(u_{n-1}, du_n) \\
&\quad \times \left[u_0 \frac{v'}{v}(x_0) + \cdots + u_n \frac{v'}{v}(x_n) \right] u_0 v(x_0) \cdots u_n v(x_n) \\
&\quad = i \sum_{n=1}^{\infty} i^n \int_{x_0 < \cdots < x_n} dx_0 \cdots dx_n \, u_0 v(x_0) \cdots u_n v(x_n) \\
&\quad \times p(du_0)\, q_{h_1}(u_0, du_1) \cdots q_{h_n}(u_{n-1}, du_n) \\
&\quad \times \left\{ u_1 \frac{q'_{h_2}}{q_{h_2}}(u_1, du_2) + u_2 \left[\frac{q'_{h_3}}{q_{h_3}}(u_2, du_3) - \frac{q'_{h_2}}{q_{h_2}}(u_1, du_2) \right] + \cdots \right. \\
&\quad \left. + u_{n-1} \left[\frac{q'_{h_n}}{q_{h_n}}(u_{n-1}, du_n) - \frac{q'_{h_{n-1}}}{q_{h_{n-1}}}(u_{n-2}, du_{n-1}) \right] - u_n \frac{q'_{h_n}}{q_{h_n}}(u_{n-1}, du_n) \right\}
\end{aligned} \tag{3}
$$

(by integrating by parts; we note $h_j = x_j - x_{j-1}$ and $q'_h = \partial q_h / \partial h$).

This equality is equivalent to the following infinite set of equations: $\forall n \in \mathbf{N}^*$, $\forall x_1 < \cdots < x_n$:

$$
\begin{aligned}
&2\, \partial_t \mathbf{E}[u(x_1) \cdots u(x_n)] \\
&\quad = \int u_1 \cdots u_n \, p(du_1)\, q_{h_2}(u_1, du_2) \cdots q_{h_n}(u_{n-1}, du_n) \\
&\quad \times \left\{ u_1 \frac{q'_{h_2}}{q_{h_2}}(u_1, du_2) + u_2 \left[\frac{q'_{h_3}}{q_{h_3}}(u_2, du_3) - \frac{q'_{h_2}}{q_{h_2}}(u_1, du_2) \right] + \cdots \right. \\
&\quad \left. + u_{n-1} \left[\frac{q'_{h_n}}{q_{h_n}}(u_{n-1}, du_n) - \frac{q'_{h_{n-1}}}{q_{h_{n-1}}}(u_{n-2}, du_{n-1}) \right] - u_n \frac{q'_{h_n}}{q_{h_n}}(u_{n-1}, du_n) \right\}.
\end{aligned} \tag{4}
$$

One then gets the evolution equations for p, q, and A by taking limits in which some of the x_is are equal. If one makes every x_i tend to x_1, the preceding set of equations gives formally, $\forall n \in \mathbf{N}^*$:

$$2 \int \partial_t p(du)\, u^n = \int p(du)(-UAU^n + U^n AU)(u)$$

where U^n denotes the function $u \mapsto u^n$.

If one makes some of the x_i's tend to x_1, and the others tend to $x_2 = x_1 + h$, one then gets $\forall n \in \mathbf{N}^*$, $\forall k < n$, $\forall x_1 \in \mathbf{R}$, $\forall h \in \mathbf{R}^{+*}$:

$$\begin{aligned} &2\, \partial_t \mathbf{E}[u(x_1)^k\, u(x_1+h)^{n-k}] \\ &\quad = \int p(du)\{-UA(U^k Q_h U^{n-k}) + U^{k+1} Q_h AU^{n-k} \\ &\qquad + U^k[A(UQ_h U^{n-k}) - AQ_h U^{n-k+1} + Q_h(U^{n-k}AU) - Q_h(UAU^{n-k})]\}(u). \end{aligned} \tag{5}$$

One then easily finds, if η is in the domain of A:

$$2 \int \partial_t p(du)\, \eta(u) = \int p(du)[-uA\eta(u) + \eta(u)\, AU(u)] \tag{6}$$

$$\begin{aligned} 2\, \partial_t Q_h \eta = {} & UAQ_h \eta + A(UQ_h \eta) - Q_h(UA\eta) \\ & - AQ_h(U\eta) + Q_h(\eta AU) - AUQ_h \eta. \end{aligned} \tag{7}$$

These two equalities sum up into one: $\forall \eta$ in the domain of A,

$$2\, \partial_t A\eta = UA^2\eta - A^2(U\eta) + A(\eta AU) - AUA\eta \tag{8}$$

or, introducing the operators M_U and M_{AU} defined as $M_U \eta(u) = u\eta(u)$ and $M_{AU}\eta(u) = AU(u)\, \eta(u)$:

$$2\partial_t A = M_U A^2 - A^2 M_U + AM_{AU} - M_{AU} A. \tag{9}$$

If this latter equality holds, one can easily check that if ${}^t\!Ap = 0$ for all time, then p verifies (6), and $Q_h = \exp(hA)$ verifies (7).

Hence a Feller statistical solution of (2) is solution of (6) and (7), which are equivalent to (9).

Conversely, it is a matter of simple algebra to check that (6) and (7) imply (2): indeed one can then write for any $x_1 < \cdots < x_n$ (recall $h_i = x_i - x_{i-1}$):

$$
\begin{aligned}
&2\,\partial_t E[u(x_1)\cdots u(x_n)]\\
&\quad = 2\partial_t \int p(du)\, M_U Q_{h_2} \cdots M_U Q_{h_n} U(u)\\
&\quad = 2\int \partial_t p(du)\, M_U Q_{h_2} \cdots M_U Q_{h_n} U(u)\\
&\qquad + 2\sum_{j=2}^{n} \int p(du)\, M_U Q_{h_2} \cdots M_U Q_{h_{j-1}} M_U\, \partial_t Q_{h_j} M_U Q_{h_{j+1}} \cdots M_U Q_{h_n} U(u)\\
&\quad = \int p(du)\, u[-A M_U Q_{h_2} M_U \cdots Q_{h_n} U(u) + Q_{h_2} M_U \cdots Q_{h_n} U(u)\, AU(u)]\\
&\qquad + \sum_{j=2}^{n} \int p(du)\, M_U Q_{h_2} \cdots M_U [M_U A Q_{h_j} \eta_j + A(U Q_{h_j} \eta_j) - Q_{h_j}(U A \eta_j)\\
&\qquad - A Q_{h_j}(U\eta_j) + Q_{h_j}(\eta_j A U) - A U A \eta_j]
\end{aligned}
\tag{10}
$$

where $\eta_j = M_U Q_{h_{j+1}} \cdots M_U Q_{h_n} U$. Many terms cancel, one gets

$$
= \int p(du)\, u \sum_{j=2}^{n} Q_{h_2} M_U \cdots Q_{h_{j-1}} M_U [M_U Q'_{h_j} - Q'_{h_j} M_U]\, \eta_j
$$

which is just one integration by parts away from (4). Then (3) follows, which is the Markovian version of (2).

Therefore, if $u(x, t)$ is a Feller process, it is a statistical solution of Burgers if and only if its infinitesimal generator is solution of (9). In some sense, the Feller assumption yields an exact closure of the infinite set (2). Of course, nothing guarantees the existence of solutions of (9), although we show later that the Brownian and white noise initial cases give formal solutions to it. Nevertheless, a close look at Bertoin's proof using Hopf–Cole[2] makes us strongly suspect that the absence of positive jumps may be essential to guarantee the existence of solutions. This would also be reasonable from a physical point of view: solutions with positive jumps are unphysical.

5. THE CASE OF LÉVY PROCESSES

We will see how one can recover formally the results of ref. 5. The initial velocity u_0 is here supposed to be a Lévy process (which means that it has independent and stationary increments) of finite variance having no negative jumps. This covers in particular the case of u_0 Brownian. Such processes are characterized by their second exponent ϕ, defined by $\forall x < y$, $\forall \lambda \in \mathbf{R}^+$:

$$E\{\exp[\lambda(u_0(y)-u_0(x))]\} = \exp[(y-x)\,\phi(\lambda)].$$

A Lévy process can be considered as a limit case of stationary Markov process (the one point distribution p is replaced with Lebesgue measure). One can also formally define an infinitesimal generator by the relations: $\forall \lambda \in \mathbf{R}^+$,

$$Ae_\lambda = \phi(\lambda)\, e_\lambda$$

where we have noted e_λ the function $u \mapsto \exp(\lambda u)$ (which of course is not in the domain of A ...). One can inject these relations into the evolution equation (9). Using $ue_\lambda(u) = \partial_\lambda e_\lambda(u)$, and $AU = \text{constant}$, one gets an evolution equation for ϕ; it turns out that this equation is also the Burgers equation:

$$2\,\partial_t \phi(\lambda) = -\partial_\lambda(\phi^2). \tag{11}$$

Carraro and Duchon[5] have checked that if ϕ_0 is the exponent of a Lévy process of finite variance with negative jumps, (11) has a smooth solution for all time $t \geqslant 0$, which is still the exponent of a homogeneous Lévy process with negative jumps.

Hence such Lévy processes are conserved by the Burgers equation. The Brownian case corresponds to $\phi_0(\lambda) = \lambda^2/2$, and this yields $\phi(\lambda, t) = (1+\lambda t - \sqrt{1+2\lambda t})/t^2$.

6. EVOLUTION EQUATION FOR THE JUMP PROCESS

The infinitesimal generator of an arbitrary Markov process can be written as the sum of three terms (see ref. 10): a diffusion term, a drift term, and a jump term:

$$Af(u) = a(u)\, f''(u) + b(u)\, f'(u) + \int n(u, dv)(f(v) - f(u)).$$

The measure $n(u, dv)$ represents the jump part of the process: it gives the mean number of jumps going from u to dv in $[x, x+dx]$, divided by dx, conditionally as $u(x) = u$. In our case, all these coefficients will of course depend on time. To write an evolution equation for n, we assume $b = 1/t$

and $a=0$ for $t>0$, and all jumps are negative. Equation (9) then yields: $\forall u>v$,

$$2\,\partial_t n(u, dv, t) = \frac{1}{t}(u-v)[\partial_u n(u, dv, t) - \partial_v n(u, dv, t)]$$
$$+\int_{-\infty}^{u} n(u, du', t)[(u-v)\, n(u', dv, t) + (v-u')\, n(u, dv, t)]$$
$$-\int_{-\infty}^{v} (u-u')\, n(v, du', t)\, n(u, dv, t). \tag{12}$$

7. THE CASE OF AN INITIAL WHITE NOISE PROCESS

Frachebourg and Martin[8] have investigated the case of an initial white noise velocity. Using the Hopf–Cole construction, they obtain explicit formulas for the laws of $u(x, t)$ and its jumps. They actually rederived results about Brownian motion with a parabolic drift that had been previously established by Groeneboom[9] out of the Burgers context. Using Frachebourg and Martin's results or Groeneboom's paper, the infinitesimal generator in the case of an initial white noise process is found to be, in the case where $\langle u_0(x)\, u_0(y)\rangle = (1/8)\,\delta(x-y)$:

$$Af(x) = \frac{1}{t} f'(x) + 4\int_{-\infty}^{x} (f(y)-f(x))(x-y)\,\frac{J(yt^{1/3})}{J(xt^{1/3})}\, I(xt^{1/3}-yt^{1/3})\, dy$$

where I and J are given by their Fourier and Laplace transforms in terms of the Airy function Ai:

$$J(u) = \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} dz\, \frac{\exp(uz)}{2^{1/3}\,\mathrm{Ai}(2^{-1/3}z)} \tag{13}$$

$$2I(u) = (2\pi u^3)^{-1/2} + \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} \exp(uz)\left(\frac{2^{2/3}\,\mathrm{Ai}'(2^{-1/3}z)}{\mathrm{Ai}(2^{-1/3}z)} + (2z)^{1/2}\right). \tag{14}$$

We have checked that the evolution equation (9) is indeed verified: it amounts to expressing convolutions like $uI * J$, $uI * uI$, $uI * uJ$ in terms of J' and I'. It can be done using relations (13) and (14) and the fact that $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$.

8. CONCLUSION

We have heuristically shown that for Feller stationary processes, Burgers equation is equivalent to an evolution equation for their infinitesimal generators. It gives strong evidence that the Feller property is conserved by Burgers equation, although we suspect that the negativity of jumps in the initial velocity should be required. Our evolution equation provides an equation for the jump process, and this might lead to other exact statistical solutions of Burgers equation. The previous exact solutions concerning an initial Brownian or white noise velocity are both particular solutions of our equation.

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